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ADDENDUM

Yang–Baxterization of algebraic solutions of reflection equation for $su(1,1)_q$ model

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Abstract. A quadratic relation between the central elements of reflection algebra for $su(1,1)_q$ model is obtained. Yang–Baxterization of algebraic solutions to reflection equation for $su(1,1)_q$ model is presented in terms of this relation.

In a previous paper [1], we studied the Yang–Baxterization of the reflection equation (RE)

$$RK_1\tilde{R}K_2 = K_2RK_1\tilde{R} \tag{1}$$

where $K_1 = K \otimes I$, $K_2 = I \otimes K$, $\tilde{R} = PRP$, and P is the permutation operator $P(x \otimes y) = y \otimes x$; namely, we presented a procedure to incorporate the parameters to R and K such that they satisfy the following parameter-dependent RE

$$R(\lambda\mu^{-1})K_1(\lambda)\tilde{R}(\lambda\mu)K_2(\mu) = K_2(\mu)R(\lambda\mu)K_1(\lambda)\tilde{R}(\lambda\mu^{-1}) \tag{2}$$

for $\check{R} = PR$ with two distinct eigenvalues. This is of significance in the factorized scattering on a half-line [2], in the quantum current algebras [3] and in the integrable models with non-periodic boundary conditions [4, 5]. In particular, we presented the Yang–Baxterization of constant solutions of RE for the $su(1,1)_q$ model. However, we failed in Yang–Baxterizing its algebraic solutions because the quantum determinant is not the central element. In this addendum, we are devoted to the Yang–Baxterization of its algebraic solution.

For convenience, we rewrite RE (1) and (2) as

$$\check{R}K_1\check{R}K_1 = K_1\check{R}K_1\check{R} \tag{3}$$

$$\check{R}(\lambda\mu^{-1})K_1(\lambda)\check{R}(\lambda\mu)K_1(\mu) = K_1(\mu)\check{R}(\lambda\mu)K_1(\lambda)\check{R}(\lambda\mu^{-1}) \tag{4}$$

in terms of $\check{R} = PR$. Here $PK_2P = K_1$ is used.

In fact, in [1] we prove that if (i) $\check{R}$ has two distinct eigenvalues t_1 and t_2 which, therefore, can be Yang–Baxterized as [6]

$$\check{R}(\lambda) = (\lambda - \lambda^{-1})\check{R} - (t_1 + t_2)\lambda I$$

and (ii) the reflection matrix K satisfies the following *Yang–Baxterization condition*

$$[\check{R}, K_1^2 + AK_1] = 0 \tag{5}$$

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where A is a central element of reflection algebra (which is an associative algebra generated by the non-commuting elements k_{ij} of the reflection matrix K [7]), then the reflection matrix can be Yang–Baxterized as

$$K(\lambda) = (\lambda - \lambda^{-1})K + A\lambda I. \tag{6}$$

We do not need the fact that K is invertical and the quantum determinant of reflection algebras is the central element. In the following we shall see that for the $su(1,1)_q$ model there exists an element which is not the central element of $\mathcal{A}_q$ but plays the role of the quantum determinant of the $sl(2)_q$ model, such that the condition (5) is satisfied, and then the algebraic solutions is Yang–Baxterized.

The presentation here is an addendum and correction of arguments in [1].

The R-matrix for the $su(1,1)_q$ model is

$$\check{R} = \begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & \omega & \\ & & & -q^{-1} \end{pmatrix} \qquad \omega = q - q^{-1}. \tag{7}$$

Letting

$$K = \begin{pmatrix} z & y \\ x & u \end{pmatrix} \tag{8}$$

and substituting $\check{R}$, K into (3) we obtain the algebraic relations of the reflection algebra $\mathcal{A}_q$ of the $su(1,1)_q$ model:

$$ux = q^2 xu \qquad [u,z] = 0 \qquad [x,z] = -q^{-1}\omega ux$$

$$uy = q^{-2} yu \qquad qxy + q^{-1}yx = \omega(uz - u^2) \qquad [y,z] = q^{-1}\omega yu \tag{9}$$

$$(q+q^{-1})x^2 = (q+q^{-1})y^2 = 0. \tag{10}$$

The case $q^2 = -1$ has been considered in paper [1]. In the following we only consider the case $q^2 \neq -1$. In this case equation (10) becomes

$$x^2 = y^2 = 0. \tag{11}$$

It is well known that the element $u - z$ is a central element. The *quantum determinant* of the K-matrix, however, is not the central element. We shall prove that there is an element which is not the central element and plays the role of quantum determinant.

According to Kulish and Sasaki [8], the reflection algebras associated with RE (3) have the following form of the central elements

$$C_n = \mathrm{Tr}(\mathcal{D}K^n) \tag{12}$$

where the matrix $\mathcal{D}$ is defined as

$$\mathcal{D} = \mathrm{Tr}_2[P((R^{t_1})^{-1})^{t_1}] \tag{13}$$

and Tr_2 denotes taking the trace in the second space.

For the case in hand, it is easy to calculate the matrix $\mathcal{D}$:

$$\mathcal{D} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{14}$$

Then from (12) we can obtain a class of central elements of $\mathcal{A}_q$. The first-order central element is the well known

$$C_1 = u - z. \tag{15}$$

The second-order central element is obtained as

$$C_2 = u^2 - z^2 + [x, y]. \tag{16}$$

Then, by direct verification using equations (9) and (11), we can prove the following:

Proposition 1. The central elements C_1 and C_2 of $\mathcal{A}_q$ satisfy the following quadratic relation

$$C_1 K^2 - C_2 K = [-zu(u - z) - zxy + yxu]I \tag{17}$$

where I is the identity matrix.

We would like to note that the element $-zu(u - z) - zxy + yxu$ in equation (17) is not the central element of $\mathcal{A}_q$. So we cannot define the inverse K^{-1} of K from (17). In comparison with the $sl(2)_q$ model, $C_1^{-1}[-zu(u-z) - zxy + yxu]$ plays the role of quantum determinant and is not the central element. However, equation (17) is essentially the *Yang–Baxterization condition*, so it plays a key role in the Yang–Baxterization of the algebraic K-matrix.

Since the $\check{R}$-matrix for the $su(1,1)_q$ model has two distinct eigenvalues q and $-q^{-1}$, the $\check{R}$ can be Yang–Baxterized as

$$\check{R}(\lambda) = (\lambda - \lambda^{-1})\check{R} - \omega\lambda I. \tag{18}$$

If we can prove that condition (5) is satisfied, then we can Yang–Baxterize the K-matrix from equation (6). In fact, condition (5) can also be modified as

$$[\check{R}, C_1 K_1^2 + B K_1] = 0 \tag{19}$$

where $B = C_1 A$ is also a central element. If we choose $B = -C_2$ then we have

$$[\check{R}, C_1 K_1^2 - C_2 K_1] = (-zu(u - z) - zxy + yxu)[\check{R}, I] = 0. \tag{20}$$

Therefore the $K(\lambda)$-matrix of $su(1,1)_q$ model is obtained as

$$K(\lambda) = (\lambda - \lambda^{-1})K - C_1^{-1} C_2 \lambda I. \tag{21}$$

Regarding x, y, z, u as the complex number, we obtain the constant solutions. Besides the identity matrix solution, there exists a non-trivial constant solution: $x = y = u = 0$, z free. Therefore the central elements take the values

$$C_1 = -z \qquad C_2 = -z^2 \tag{22}$$

and equation (21) reduces to

$$K(\lambda) = (\lambda - \lambda^{-1})K - z\lambda I \tag{23}$$

which is same as the result in [1].

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